Linear Algebra I

Mid Term Examination

Instructions: All questions carry ten marks. Vector spaces are assumed to be finite dimensional.

1. Give examples of symmetric matrices A and B such that $AB \neq BA$.

2. Let A be a square matrix such that the system of equations $AX = B$ has a unique solution for *some vector* B. Then show that for every vector C, there exists a unique solution for the system $AX = C$.

3. Show that the set of Trace zero real matrices is a subspace of the vector space of all real matrices of order n. (You may assume that the set of matrices of order n is a vector space over real numbers under usual addition and scalar multiplication.)

4. Prove or disprove: Any set S of a vector space V over a field F contains a linearly independent subset $T \subset S$ such that $\text{Span}(T) = \text{Span}(S)$.

5. Prove that given a maximal linearly independent subset S of V, any vector can be written in a unique way (up to a reordering elements of S) as a linear combination of elements of S.

6. Let W be a subspace of $\mathbb{R}^n$. Prove that there exists a subspace W_1 such that

   (a) $W \cap W_1 = 0$, and

   (b) every vector v of $\mathbb{R}^n$ can be written uniquely as $v = w + w_1$ with $w \in W$ and $\mathrm{w}_1 \in W_1$.